The missing ingredient in the linear approximation of the λ-calculus

and other autobiographical stuff

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The main story

APPROXIMATING THE λ-CALCULUS?

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to approximate the total information generated by M
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("The total information generated by M": the **Böhm tree** of M.)

"Syntactic" approximation theorem:

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The continuous approximation

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$= \bigsqcup \Big\{$  $\beta\bot$-normal $\lambda\bot$-term $\Big|\; M \longrightarrow_{\beta}^{*}$  $\Big\}.$

"Commutation" theorem (Ehrhard-Regnier'06):

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... where $\mathcal{T} : \Lambda_\perp \to ?$ is defined by

$$\begin{aligned}
\mathcal{T}(x) &:= x \\
\mathcal{T}(\lambda x.M) &:= \lambda x.\mathcal{T}(M) \\
\mathcal{T}(MN) &:= \mathcal{T}(M) \sum_{n\in\mathbb{N}} \frac{1}{n!}\mathcal{T}(N)^n \\
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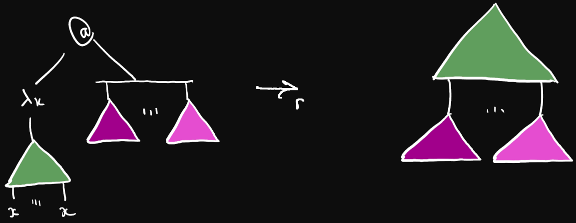
We need: multisets as arguments, sums of terms.

Resource terms:

$$s, t, \ldots \quad := \quad x \quad | \quad \lambda x.s \quad | \quad (s)\,[t_1, \ldots, t_n].$$

Resource reduction, featuring a multilinear substitution:



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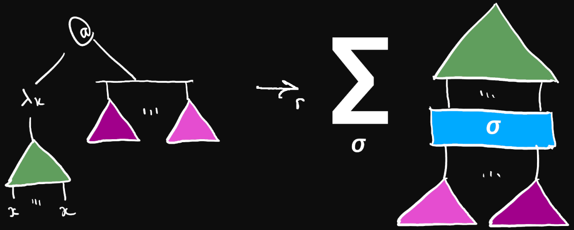

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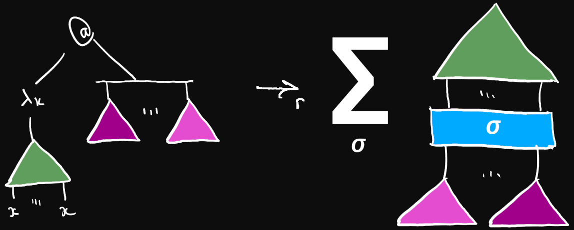
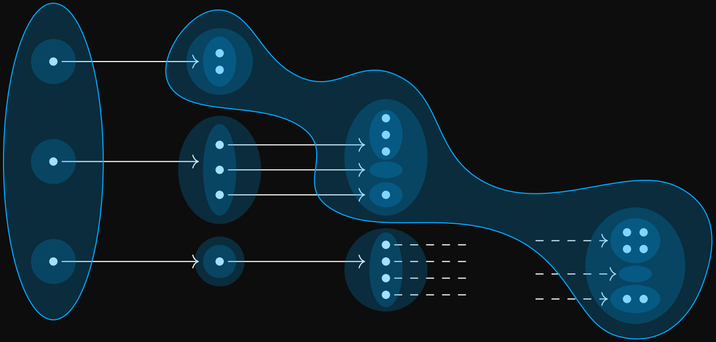


Excellent properties (confluence, normalisation)!

Finally, $\mathbf{S} \longrightarrow\!\!\!\!\rightarrow_r \mathbf{T}$ denotes the pointwise reduction (through $\longrightarrow_r^*$) of possibly infinite sums of resource terms.



nf($\mathbf{S}$) is the pointwise normal form of $\mathbf{S}$.

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$$\mathcal{T}(\mathrm{BT}(M)) = \mathrm{nf}(\mathcal{T}(M)).$$

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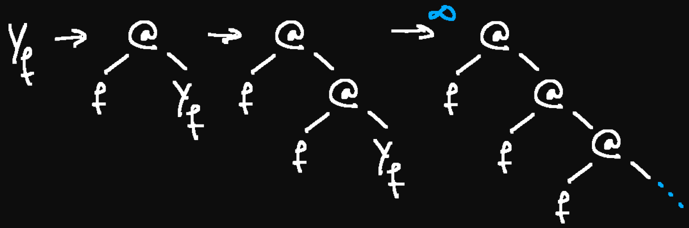
Here, a "syntactic" motivation:
to approximate the total dynamics ("information flow") of *M*
using pieces of finite dynamics ("finite information flows")

Theorem (Vaux'17):
If $M \longrightarrow^*_{\beta\bot} N$ then $\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(N)$.

This is not enough: we can't talk about BT(M)...

- We still don't know what $\mathcal{T}(\mathrm{BT}(M))$ is.
- BT(M) may be infinitely far from M.

We want possibly infinite terms and reductions of a certain shape:



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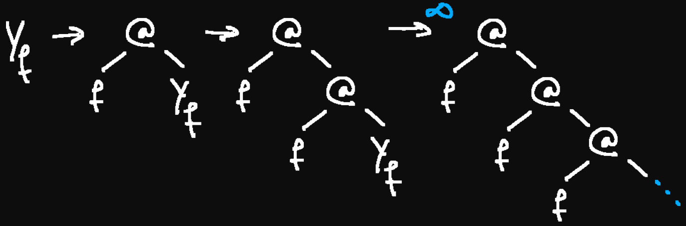

Theorem (Kennaway *et al.*'97):
$\longrightarrow^{001}_{\beta\bot}$ is confluent, and the unique normal form of any $M \in \Lambda^{001}_{\bot}$ through $\longrightarrow^{001}_{\beta\bot}$ is $\mathrm{BT}(M)$.

One approximation theorem to rule them all

$\mathcal{T} : \Lambda_{\perp}^{001} \to \mathbb{S}^{\Lambda_{\mathrm{r}}}$ is defined (almost) as on finite terms (!).

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"Simulation" theorem (C.-V.A.'23, C.'24):

If $M \longrightarrow_{\beta\perp}^{001} N$ then $\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(N)$.

All the previous ones are easy consequences:

$$\begin{array}{ccc} \text{simulation theorem} & \Longrightarrow & \text{commutation theorem} \\ \Downarrow & & \Downarrow \\ \text{confluence of } \longrightarrow_{\beta\perp}^{001} & \Longrightarrow & \text{syntactic approx. theorem} \end{array}$$

(Arguably) easier proofs, in a unified setting

Corollary

M has a HNF through $\longrightarrow_\beta^*$ or $\longrightarrow_\beta^{001}$
iff the head reduction strategy terminates on M
iff $\mathrm{nf}(\mathcal{T}(M)) \neq 0$.

Corollary

The Genericity lemma.

Corollary

$\mathrm{BT} : \Lambda_\perp^{001} \to \Lambda_\perp^{001}$ is Scott-continuous.

Corollary

$\mathcal{B}$ is a λ-theory

The bonuses

Let's be lazy:

head normal forms	$\to$	weak head normal forms
Böhm trees	$\to$	Lévy-Longo trees
$\Lambda_{\perp}^{001}$	$\to$	$\Lambda_{\perp}^{101}$
$\longrightarrow_{\beta\perp}^{001}$	$\to$	$\longrightarrow_{\beta\perp}^{101}$

Example: $\mathsf{Y}_{\mathsf{K}} \longrightarrow_{\beta}^{*} \lambda x.\mathsf{Y}_{\mathsf{K}}$ is such that:

$$\mathrm{BT}(\mathsf{Y}_{\mathsf{K}}) = \perp \qquad \mathrm{LLT}(\mathsf{Y}_{\mathsf{K}}) = \lambda x_0.\lambda x_1.\lambda x_2.\ldots$$

What I've been working on (1)

The **lazy resource λ-calculus**:

$$s, t, \ldots \quad := \quad x \quad | \quad \lambda x.s \quad | \quad \mathbb{0} \quad | \quad (s)\,[t_1, \ldots, t_n],$$

with $\quad (\mathbb{0})\,\bar{t} \longrightarrow_{\mathrm{r}} 0 \quad$ and $\quad \ell\mathcal{T}(\lambda x.M) := \lambda x.\ell\mathcal{T}(M) + \mathbb{0}.$

Theorem (simulation)

If $M \longrightarrow^{101}_{\beta\perp} N$ then $\ell\mathcal{T}(M) \twoheadrightarrow_{\ell\mathrm{r}} \ell\mathcal{T}(N)$.

Corollary (commutation)

$\mathrm{nf}(\ell\mathcal{T}(M)) = \ell\mathcal{T}(\mathrm{LLT}(M))$.

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If $M \longrightarrow_{\beta\bot}^{101} N$ then $\ell\mathcal{T}(M) \twoheadrightarrow_{\ell\mathrm{r}} \ell\mathcal{T}(N)$.

Corollary (commutation)

$\mathrm{nf}(\ell\mathcal{T}(M)) = \ell\mathcal{T}(\mathrm{LLT}(M))$.

Theorem (Severi-de Vries'05)

Only BT and LLT are Scott-continuous.

WHAT I'VE BEEN WORKING ON (2)

001-infinitary λ-terms, definition 1:

$$\frac{x \in \mathcal{V}}{x \in \Lambda_{\perp}^{001}} \qquad \frac{x \in \mathcal{V} \qquad M \in \Lambda_{\perp}^{001}}{\lambda x.M \in \Lambda_{\perp}^{001}} \qquad \frac{}{\perp \in \Lambda_{\perp}^{001}}$$

$$\frac{M \in \Lambda_{\perp}^{001} \qquad \triangleright N \in \Lambda_{\perp}^{001}}{MN \in \Lambda_{\perp}^{001}} \qquad \frac{N \in \Lambda_{\perp}^{001}}{\overline{\triangleright N \in \Lambda_{\perp}^{001}}}$$

and we quotient by α-equivalence.

001-infinitary λ-terms, definition 2:

$$\Lambda_{\perp}^{001} := \nu Y.\mu X.\mathcal{V} + (\mathcal{V} \times X) + (X \times Y) + \perp$$

in the category of nominal sets (C., FICS'24).

Is the linear approximation conservative?

Counterexample

There are $A, \bar{A} \in \Lambda^{001}$ such that $\mathcal{T}(A) \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(\bar{A})$ but there is no reduction $A \longrightarrow_{\beta}^{001} \bar{A}$.

(C. and V.A., preprint)

What I'm working on:

- non-wellfounded proofs (with Alexis)
  - coinductive presentation of cut-elimination
  - model checking as circular proof search
  - natural deduction style?
- refined or extended approximations (with Giulio, Alexis)
  - Böhm trees with data on infinite branches
  - $\Lambda\mu$
- topological dynamics (with Marseille people)

What I'd like to work on:

- links between all that and automata/languages
  (with Alexis, Paul-André, and?)

Thanks for your attention!